Olin College of Engineering
Mathematical and Physical Foundations in Engineering II
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Self-Taught Assignment
Spring 2004
Due: April 23, 2004

The most practical capacitors used today are the ones that offer large capacitance in small packaging. To manufacture such capacitors, two flat, thin strips of conducting material are separated by a dielectric material (polyethylene, for example). These are then rolled up and placed into a cylindrical case (see Figure 1a). Such electronic components are often referred to as cylindrical capacitors. The goal of this activity is to determine how the overall capacitance of this device is affected by nonuniformities within a conducting plate. In what follows, we will refer to a capacitor as "ideal" if it does not have any nonuniformities. Assume that you are given a capacitor in which both the conducting plates and dielectric sheets have thickness 5 μm. Take the radius of this cylindrical capacitor to be R cm and its length to be L cm.

You may work with your teammates, however the final report must be your individual interpretation of the solution to the problem. Your grade on this assignment is based only on this report. In working through the problem, you must address the questions below in a clear and cohesive fashion.

1. There are two ways to determine the overall capacitance of an ideal capacitor.

   (a) The first method involves calculating the total capacitance by first considering a small piece of an ideal capacitor of capacitance dC (Figure 1b). Integrating this capacitance along the spiral path will yield an estimate for the capacitance of this electronic device. Determine the overall capacitance of a cylindrical capacitor of radius $R = 2$ cm and length $L = 1$ cm in terms of the total number of revolutions, N, needed to roll up the plates to form the capacitor.

   (b) The second method involves calculating the capacitance by modeling the cylindrical (i.e. spiral) capacitor using many sets of two concentric annular capacitors that share one conducting plate. Figure 3 depicts two capacitors C_1 and C_2 whose common plate is at $r = r_o$. Determine the effective capacitance of the system of these two capacitors. Assume that the separation between each pair of plates is given by $\Delta r \approx 5\mu$m. Next, find a mathematical expression for the sum of *all* pairs of capacitors necessary to form the final cylindrical capacitors. (HINT: You can assume as a first approximation that $\Delta r \ll r_o$, and look at the leading-order term in a Maclaurin series expansion).

   Compare the result in the second method with the leading-order behavior found in the first method if $N \gg 1$. What are the pros and cons of the two approaches to the approximation of the overall capacitance?

2. In the remaining parts to this problem, we shall consider extensions of the concentric-cylinder model. The outer plate at $r = r_o + \Delta r$ has the same electric potential value as the inner plate at $r = r_o - \Delta r$, or

$$V_o(r_o - \Delta r, \theta) = V(r_o + \Delta r, \theta) = V_1 \ .$$

The plate at $r = r_o$, however, has nonuniformities along it so that its electric potential is given by

$$V(r_o, \theta) = V_o \left\{ 1 + \frac{\delta}{\Delta r} \cos^2(m\theta) \right\} \ ,$$

where V_o is the anticipated voltage of the plate, δ is the degree of nonuniformity in the plate, and an integer m is the variability of potential within the plate. To find the total capacitance, the two annular capacitors must be appropriately taken into account.

(a) For the case when $\delta \neq 0$, assume that the potential in each annular region to be of the form

$$V_+(r, \theta) = a_o(r) + a_1(r) \cos\theta + a_2(r) \cos 2\,\theta + \cdots a_n(r) \cos n\,\theta + \cdots \tag{1}$$

$$V_-(r, \theta) = b_o(r) + b_1(r) \cos\theta + b_2(r) \cos 2\,\theta + \cdots b_n(r) \cos n\,\theta + \cdots \ , \tag{2}$$

where $V_+(r, \theta)$ is the voltage in the region $r_o < r < r_o + \Delta r$ while $V_-(r, \theta)$ is the voltage in the region $r_o - \Delta r < r < r_o$. Find the differential equations for each of the $a_i(r), b_i(r)$ from the Laplace equation for the voltage potential (i.e. Gauss's law):

$$\frac{1}{r} \frac{\partial}{\partial r} \left\{ r \frac{\partial V}{\partial r} \right\} + \frac{1}{r^2} \frac{\partial^2 V}{\partial \theta^2} = 0 \ .$$

Note that you have to find the boundary conditions for $a_i(r), b_i(r)$ from the assumptions on the voltage (1), (2).

(b) Solve the differential equations for $a_i(r), b_i(r)$ above.

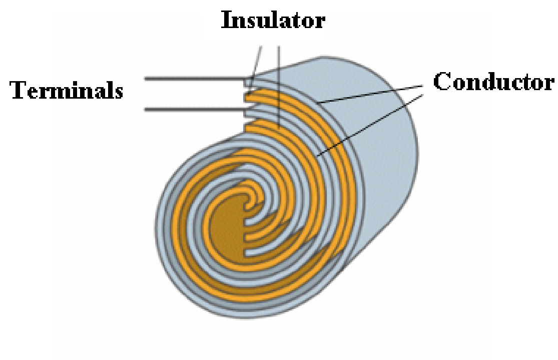
(c) Find the total charge on each plate. To find this, use Gauss's Law in integral form.

(d) Find the capacitance of the model system in Figure 2, assuming that the voltage difference is given by the ideal case $\delta = 0$, but the charge Q is given by the calculation above.

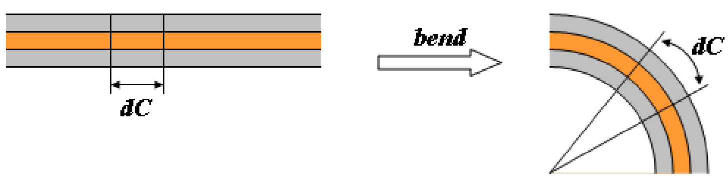
(e) Find the capacitance of the whole capacitor for $\delta \neq 0$.

(f) Compare the flawed capacitor to the flawless capacitor for values of $\delta = 0.001, 0.01, 0.1\mu$m with $V_1 = 24\,V$ and $V_o = 12\,V$. Assume that $R = 1\,$cm and $L = 3\,$cm.

The format of the report is given in the Project Format Page, and a sample paper can be found at this link.



(a)



(b)

Figure 1: (a) Fabrication technique for the development of a cylindrical capacitor. A stratified sheet of aluminium foil and polyethylene are rolled to form a cylindrical shape. (b) Description of capacitor element dC.

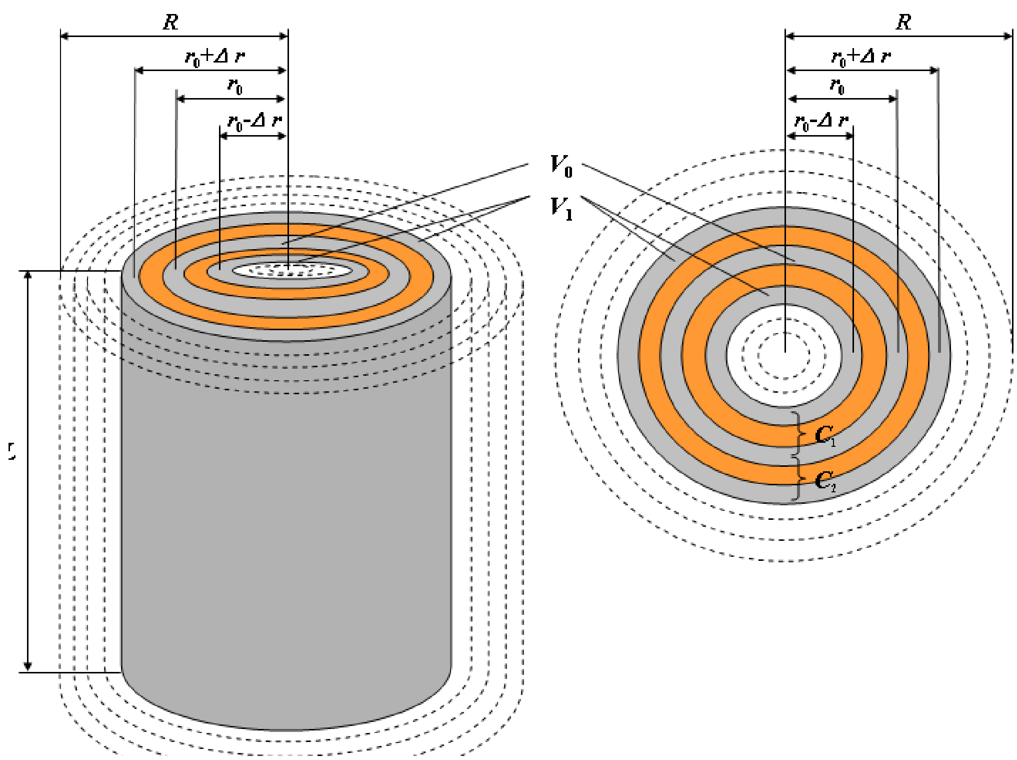


Figure 2: Model geometry to approximate the capacitance of the rolled capacitor. This sample pair is repeated thousands of times in a concentric fashion.